CONTENTS

Chapter - 1

Matrices

1.1 Introduction

Matrices are the mathematical tools, which can be used for solving the problems arising in engineering, physics, economics, statistics and so on. This chapter deals with various types, operations & properties of matrices and also with the task of finding rank of matrices.

1.2 Recapitulation of Matrices

A rectangular arrangement of numbers along the rows and columns is called a **matrix**. The array will be enclosed between the brackets ( ) or [ ]. If there are *m* number of rows and *n* number of columns then the matrix is said to be of the **order** $m \times n$ (read as *m* by *n*). The upper case letters such as *A*, *B*, *C*, *M*, *N*, *X*, *Y* etc. are used to denote a matrix.

For example,

$$A = \begin{bmatrix} 7 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 3 & 4 \end{bmatrix}_{3\times 3}, M = \begin{pmatrix} 5 & 1 \\ 7 & 3 \\ 2 & 4 \end{pmatrix}_{3\times 2}, \quad X = \begin{bmatrix} 2 & 4 & 5 \\ 9 & 0 & 1 \end{bmatrix}_{2\times 3}$$

In general,

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}_{m\times n}$$

1.3 Types of matrices

1. A matrix in which every element is zero is called a **zero matrix**, or **null matrix** or **void matrix**.

For example,

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$$

2. A matrix containing only one row is called a **row matrix**

   For example,

$$R = \begin{bmatrix} 2 & 5 & 7 & 1 \end{bmatrix}_{1\times 4}$$

3. A matrix containing only one column is called a **column matrix.**

   For example,

$$C = \begin{bmatrix} 7 \\ 9 \\ 3 \end{bmatrix}_{3\times 1}$$

4. A matrix having equal number of rows & columns is called a **square matrix**

   For example,

$$X = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}_{2\times 2}, \quad M = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 4 & 5 \\ 7 & 9 & 8 \end{bmatrix}_{3\times 3}$$

Note: The elements along left-top corner to right - bottom corner constitute a **principal diagonal.**

5. A square matrix having non-zero principal diagonal elements & all off - diagonal elements as zero is called a **diagonal matrix**

   For example,

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}_{3\times 3}$$

6. A diagonal matrix with a same element along the principal diagonal is called a **scalar matrix**

   For example,

$$S = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

7. A scalar matrix with the diagonal elements equal to 1 is called a **unit matrix or an identit matrix**

For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ \& so on.}$$

8. **Equal matrices :** Two matrices are said to be equal if they are of same order and their corresponding elements are equal.

For example ,

$$\text{If } X = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix},\ Y = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \text{ then}$$

$X = Y$ if and only if $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$.

1.4 Matrix operations

1. **Addition :** If A & B are the matrices of same order $m \times n$, then $A + B$ is also a matrix of order $m \times n$, which is obtained by adding the corresponding elements of A & B.

For example ,

$$\text{If } A = \begin{bmatrix} 2 & 6 & 3 \\ 1 & 3 & 9 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 2 & 0 \end{bmatrix} \text{ then } A + B = \begin{bmatrix} 3 & 8 & 7 \\ 7 & 5 & 9 \end{bmatrix}$$

Note:

1. $A + B = B + A$ (commutative)
2. $A + (B + C) = (A + B) + C$ (associative)
3. $A + 0 = 0 + A = A, A + B = 0 \Rightarrow A = -B$, (0 being a null matrix).

2. **Scalar multiplication:** If A is a matrix of order $m \times n$, & k is any constant then kA is also matrix of order $m \times n$, which is obtained by multiplying every element of A by k.

For example,

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \end{bmatrix} \text{ then } kA = \begin{bmatrix} k & 2k & 3k \\ 2k & 0 & k \end{bmatrix}$$

3. **Matrix multiplication:** Two matrices A & B can be multiplied only when the number of columns of A is equal to the number of rows of B. i.e., A is of order $m \times n$ & B is of order $n \times p$ then their product AB is a matrix of order $m \times p$, which is defined as given below.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times3} \ \& \ B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times2} \text{ then}$$

$$AB = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}+b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{33} \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Note:

1. The i^{th} row & j^{th} column element of AB (i.e., c_{ij}) is obtained by multiplying the elemen row of A with the corresponding the elements of j^{th} column of B & then adding these pr

   i.e., $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

2. $AB \neq BA$ (i.e., matrix multiplication is not commutative).

1.5 Transpose of a Matrix

A matrix obtained by interchanging rows and columns of a given matrix is called **tran** of that matrix.

The transpose of a matrix $A = [a_{ij}]$ is denoted by A^T or $A' = [a_{ji}]$.

For example,

$$\text{If} \quad A = \begin{bmatrix} 3 & 7 & 2 & 1 \\ 1 & 2 & 5 & 9 \\ 6 & 2 & 3 & 4 \end{bmatrix}_{3\times 4} \quad \text{then } A' = \begin{bmatrix} 3 & 1 & 6 \\ 7 & 2 & 2 \\ 2 & 5 & 3 \\ 1 & 9 & 4 \end{bmatrix}_{4\times 3}$$

Note:

1. The $(j, i)^{th}$ element i.e., j^{th} – row, i^{th} – column element of A' is the $(i, j)^{th}$ element of A.
2. If A is of order $m \times n$ then A' will be of order $n \times m$.
3. $(A')' = A$, for any matrix A.
4. $(A+B)' = A' + B'$, for any matrices A and B.
5. $(kA)' = kA'$, k being any scalar.
6. $(AB)' = B'A'$ (Reversal law).

1.6 Symmetric matrix

Any square matrix $A = [a_{ij}]$ is said to be symmetric if

$$a_{ij} = a_{ji} \quad \forall\, i, j$$

For example,

$$\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}_{2\times 2}, \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 7 \\ 2 & 7 & 6 \end{bmatrix}_{3\times 3}, \begin{bmatrix} a & p & q \\ p & b & r \\ q & r & c \end{bmatrix}_{3\times 3} \text{ etc are } \textbf{symmetric}.$$

Note : Every diagonal matrix is symmetric.

Theorem : The necessary and sufficient condition for the square matrix A to be symmetric is that $A = A'$.

Proof :

First suppose A is symmetric.

$\Rightarrow \quad a_{ij} = a_{ji}, \forall ij$ ---- (1)

Now to prove $A = A'$ or $A' = A$

Consider $(i, j)^{\text{th}}$ element of $A' = (j, i)^{\text{th}}$ element of A. (by def.)

$= a_{ji}$

$= a_{ij}$ (by (1))

$= (i, j)^{\text{th}}$ element of $A, \forall ij$.

$\Rightarrow$ The corresponding elements of A' and A are equal.

$\Rightarrow \quad A' = A.$

$\Rightarrow \quad A = A'.$

Conversely suppose $A' = A$ ---- (2)

Consider now, $\forall i, j$,

$a_{ij} = (i, j)^{\text{th}}$ element of A

$= (i, j)^{\text{th}}$ element of A' (by (2))

$= (j, i)^{\text{th}}$ element of A (by def.)

i.e., $a_{ij} = a_{ji}, \quad \forall i, j$

Hence A is symmetric.

1.7 Skew - Symmetric matrix

Any square matrix $A = [a_{ij}]$ is said to be **skew symmetric** if

$$a_{ij} = -a_{ji}, \quad \forall i, j.$$

Theorem : Every diagonal element of a skew-symmetric matrix is zero.

Proof : Let $A = [a_{ij}]$ be skew-symmetric.

$\Rightarrow \quad a_{ij} = -a_{ji}, \quad \forall i, j.$ ---- (1)

If $i = j$, we have

$a_{ii} = -a_{ii}$ (by (1))

$\Rightarrow a_{ii} + a_{ii} = 0$

$\Rightarrow \quad 2a_{ii} = 0$

$\Rightarrow \quad a_{ii} = 0, \quad \forall i$

$\Rightarrow$ Every diagonal element of A is zero.

Theorem : The necessary and sufficient condition for a square matrix A to be skew-symmetric that $A' = -A$.

Proof :

First suppose, $A = [a_{ij}]$ be skew-symmetric

$\Rightarrow \quad a_{ij} = -a_{ji}, \quad \forall\, i, j.$

$\Rightarrow \quad -a_{ij} = a_{ji}, \quad \forall\, i, j.$ ---- (1

To prove A' = A, consider $(i, j)^{th}$ element of $A' = (j, i)^{th}$ element of A,

$= a_{ji}$,

$= -a_{ij}$, (by (1))

$= (i, j)^{th}$ element of $-A$, $\forall\, i, j$,

i.e., the corresponding elts of A' and $-A$ are equ

$\therefore \quad A' = -A$

Conversely suppose that

$A' = -A$ or $-A' = A$ or $A = -A'$. ---- (2

To prove A is skew-sym.

$a_{ij} = (i, j)^{th}$ element of $A = (i, j)^{th}$ element of $(-A')$. (by (2)

$= -\{(i, j)^{th}$ element of $A'\}$.

$= -\{(j, i)^{th}$ element of $A\}$ (by def.)

$= -a_{ji}, \ \forall\, i, j.$

i.e., $\quad a_{ij} = -a_{ji}, \ \forall\, i, j$

Thus A is skew symmetric.

1.8 Conjugate of a matrix

A matrix obtained from a given matrix (with complex numbers as the elements) by replacing its elements by the corresponding complex cojugates is called the **conjugate** of the given matrix.

If $A = [a_{ij}]_{m \times n}$ is a given matrix, its **conjugate** is denoted by $\overline{A} = [\overline{a_{ij}}]_{m \times n}$. Here a_{ij} are complex numbers of the form $\alpha + i\beta$ and thus $\overline{a_{ij}} = \alpha - i\beta$, the complex conjugate of a_{ij}.

For example,

$$A = \begin{bmatrix} 1+i & i & 2+3i \\ 0 & 4-i & 7-2i \\ 3 & -5i & 7i \end{bmatrix} \text{ then } \overline{A} = \begin{bmatrix} 1-i & -i & 2-3i \\ 0 & 4+i & 7+2i \\ 3 & 5i & -7i \end{bmatrix}$$

Note :

1. $\overline{\left(\overline{A}\right)} = A$
2. $\overline{(A+B)} = \overline{A} + \overline{B}$
3. $\overline{AB} = \overline{A} \cdot \overline{B}$
4. $\overline{kA} = \overline{k} \cdot \overline{A}$ where k is any complex number.

1.9 Tranjugate of a matrix

The transpose of a conjugate of a given matrix A is called the **transposed conjugate** of A or **tranjugate** of A. It is denoted by A^*.

$$\therefore \quad A^* = \left(\overline{A}\right)' \text{ or } \left(\overline{A'}\right).$$

For example,

$$A = \begin{bmatrix} 3i & 2+i & 7 \\ 1-i & 4i & -5i \\ 6+2i & -5 & 1+2i \end{bmatrix}_{3\times3} \quad \text{then } A^* = \begin{bmatrix} -3i & 1+i & 6-2i \\ 2-i & -4i & -5 \\ 7 & +5i & 1-2i \end{bmatrix}$$

Note :

1. $(A^*)^* = A$
2. $(A+B)^* = A^* + B^*$
3. $(AB)^* = B^* A^*$ (Reversal law)
4. $(kA)^* = \overline{k}\, A^*$, where k is any complex number.

1.10 Hermitian matrix

A square matrix $A = \left[a_{ij}\right]$ is said to be Hermitian matrix if

$$a_{ij} = \overline{a_{ji}}, \forall i, j.$$

Theorem : Every diagonal element of a Hermitian matrix is purely real.

Proof : Let $A = \left[a_{ij}\right]$ be Hermitian.

$$\therefore \quad a_{ij} = \overline{a_{ji}}, \forall i, j$$

Now for the diagonal elements, $j = i$ so that

$$a_{ii} = \overline{a_{ii}}$$

$$\left(\begin{array}{l} \because \alpha + i\beta = \alpha - i\beta \\ \Rightarrow 2i\beta = 0 \Rightarrow \beta = 0 \\ \Rightarrow \alpha + i0 = \alpha, \text{ real} \end{array}\right)$$

$\Rightarrow$ a_{ii} is purely real

$\therefore$ Every diagonal element is real.

For example,

$$\begin{bmatrix} 2 & 3-i \\ 3+i & 4 \end{bmatrix}, \begin{bmatrix} 3 & 2i & 1+i \\ -2i & 5 & -7i \\ 1-i & 7i & 2 \end{bmatrix} \text{ are Hermitian matrices.}$$

Theorem : The matrix A is Hermitian if and only if $A = A^*$.

Proof :

First suppose A is Hermitian.

$$\Rightarrow \quad a_{ij} = \overline{a_{ji}}, \forall i, j \qquad \text{---- (1)}$$

To prove $A = A^*$ or $A^* = A$

Consider, $(i, j)^{\text{th}}$ element of $A^* = (i, j)^{\text{th}}$ element of $(\overline{A})'$

$= (j, i)^{\text{th}}$ element of $\overline{A}$

$=$ complex conjugate of $(j, i)^{\text{th}}$ element of A.

$= \overline{a_{ji}}$

$= a_{ij}$ (by (1))

i.e., $(i, j)^{\text{th}}$ element of $A^* = (i, j)^{\text{th}}$ element of A, $\forall i, j$.

$\Rightarrow$ The corresponding elements of A and A^* are equal.

$\Rightarrow$ $A = A^*$.

Conversely suppose $A = A^*$

$$\Rightarrow \quad A = (\overline{A})' \qquad \text{---- (2)}$$

$$\Rightarrow \quad (\overline{A})' = A$$

$$\Rightarrow \quad \overline{(\overline{A})'} = \overline{A}$$

$$\Rightarrow \quad \left[\overline{(\overline{A})}\right]' = \overline{A}$$

$$\Rightarrow \quad A' = \overline{A} \qquad \text{---- (3)}$$

Now consider

$$a_{ij} = (i, j)^{\text{th}} \text{ element of } A$$

$= (j,i)^{\text{th}}$ element of A' (by definition)

$= (j,i)^{\text{th}}$ element of $\overline{A}$ (by (3))

$=$ complex conjugate of $(j,i)^{\text{th}}$ element of A (by definition)

i.e., $a_{ij} = \overline{a_{ji}},\ \forall i,j.$

Hence A is Hermitian.

1.11 Skew-Hermitian matrix

A square matrix $A=\left[a_{ij}\right]$ is said to be skew-Hermitian if

$$a_{ij} = -\overline{a_{ji}},\ \forall i,j$$

Theorem : Every diagonal element of a skew-Hermitian matrix is either purely imaginary or zero.

Proof : Let $A=\left[a_{ij}\right]$ be skew-Hermitian

$$a_{ij} = -\overline{a_{ji}},\ \forall i,j.$$

For diagonal element, $j=i$, so that

$$a_{ii} = -\overline{a_{ii}}$$

$$\Rightarrow\ a_{ii}+\overline{a_{ii}} = 0.$$

$\Rightarrow\ a_{ii}$ is either purely imaginary or zero. $\left(\begin{array}{r}\because (\alpha+i\beta)+(\alpha-i\beta)=0\\ \Rightarrow\ 2\alpha=0\Rightarrow\alpha=0\end{array}\right)$

heorem : The matrix A is skew-Hermitian if and only if $A^* = -A$

roof : First suppose that the matrix $A=\left[a_{ij}\right]$ is skew-Hermitian

$\Rightarrow\ a_{ij} = -\overline{a_{ji}}$

$= -(j,i)^{\text{th}}$ element of $\overline{A}$

$= (j,i)^{\text{th}}$ element of $\left(-\overline{A}\right)$ ---- (1)

Also, $a_{ij} = (i,j)^{\text{th}}$ element of A

$= (j,i)^{\text{th}}$ element of A' ---- (2)

From (1) and (2)

$(j,i)^{\text{th}}$ element of $\left(-\overline{A}\right) = (j,i)^{\text{th}}$ element of A', $\forall i,j.$

$\Rightarrow\ \left(-\overline{A}\right) = A'$ ($\because$ corresponding elements are equal)

$\Rightarrow\ \overline{\left(-\overline{A}\right)} = \overline{A'}$

$$\Rightarrow \quad -\overline{(\overline{A})} = A^*$$

$$\Rightarrow \quad -A = A^* \text{ or } A^* = -A$$

Conversly suppose that $A^* = -A$

$$\Rightarrow \quad \overline{A}' = -A$$

$$\Rightarrow \quad \overline{(\overline{A}')} = -\overline{A}$$

$$\Rightarrow \quad A' = -\overline{A}$$

i.e., $(j, i)^{th}$ element of $A' = (j, i)^{th}$ element of $(-\overline{A})$, $\forall i, j$.

$\Rightarrow$ $(i, j)^{th}$ element of $A = -(j, i)^{th}$ element of $\overline{A}$, $\forall i, j$.

$$\Rightarrow \quad a_{ij} = -\overline{a_{ji}}, \ \forall i, j$$

Hence A is Skew-Hermitian.

WORKED EXAMPLES

Example 1:

If A, B are symmetric, show that $(A + B)$ is also symmetric.

Solution :

Let A and B be the symmetric matrices

$\Rightarrow \quad A' = A$ ---- (1

and $\quad B' = B$ ---- (2

Consider

$(A + B)' = A' + B'$ (by property of transpose

$\qquad = A + B$ (by (1) and (2)

i.e., $(A + B)' = (A + B)$

Hence $(A + B)$ is a symmetric matrix.

Example 2:

If A is a square matrix, show that $(A\,A)'$ and $(A'\,A)$ are symmetric.

Solution :

Consider

$(A\,A')' = (A')'\,A'$ (Reversal law

$\qquad = A \cdot A'$ $(\because\ (A')' = A$

i.e., $(A A')' = A A'$

$\Rightarrow$ AA' is symmetric.

Also, consider $(A' A)' = A' \cdot (A')'$ (Reversal law)

$= A' \cdot A$

i.e., $(A' A)' = A' \cdot A$

$\Rightarrow$ $A' A$ is also symmetric.

***Example* 3:**

If A is symmetric, show that A^2 and A^n (where n is any positive integer) are symmetric.

Solution :

Since A is symmetric $A' = A$

To show that A^2 is symmetric, consider

$$\begin{aligned}(A^2)' &= (AA)' = A' \cdot A' \\ &= (A')^2 \\ &= A^2 \qquad (\because A' = A)\end{aligned}$$

i.e., $(A^2)' = A^2 \Rightarrow A^2$ is symmetric.

Now to show that A^n is symmetric, consider

$$\begin{aligned}(A^n)' &= (A \cdot A \cdot A \cdot \text{............} n \text{ times})' \\ &= A' \cdot A' \cdot A' \text{............} n \text{ times} \qquad \text{(general reversal law)} \\ &= A \cdot A \cdot A \cdot \text{............} n \text{ times} \qquad (A' = A)\end{aligned}$$

i.e., $(A^n)' = A^n$

Hence A^n is symmetric.

***Example* 4:**

If A and B are symmetric then AB is also symmetric if and only if $AB = BA$.

Solution :

Since A and B are symmetric

$\Rightarrow$ $A' = A$ ---- (1)

and $B' = B$ ---- (2)

First suppose that AB is symmetric.

$$\begin{aligned}AB &= (AB)' \\ &= B'A' \qquad \text{(Reversal law)}\end{aligned}$$

i.e., $AB = BA$ (By (1) and (2))

Conversely suppose that,

$AB = BA$ ---- (3)

Consider $(AB)' = B'A'$ (Reversal law)

$= BA$ (by (2) and (1))

i.e., $(AB)' = AB$ (by (3))

$\Rightarrow$ AB is symmetric.

Example 5:

If A, B are skew-symmetric, show that $(A + B)$ is skew-symmetric.

Proof : Since A and B are skew-symmetric

$A' = -A$ and $B' = -B$ ---- (1)

To show that $(A + B)$ is skew-symmetric,

Consider $(A + B)' = A' + B'$

$= -A - B$ (by (1))

i.e., $(A + B)' = -(A + B)$

$\Rightarrow$ $(A + B)$ is also skew-symmetric.

Example 6:

For any square matrix A, show that

(i) $A + A'$ is symmetric.

(ii) $A - A'$ is skew-symmetric.

Proof : (i) To show that $A + A'$ is symmetric,

Consider $(A + A')' = A' + (A')'$ $(\because (A+B)' = A' + B')$

$= A' + A$ $(\because (A')' = A)$

i.e., $(A + A')' = A + A'$ $(\because A + B = B + A)$

$\Rightarrow$ $(A + A')$ is symmetric.

(ii) To show that $(A - A')$ is skew-symmetric.

Consider,

$(A - A')' = A' - (A')'$ $(\because (A - B)' = A' - B')$

$= A' - A$ $(\because (A')' = A)$

i.e., $(A - A')' = -(A - A')$ (taking minus sign common)

$\Rightarrow$ $(A - A')$ is skew-symmetric.

Example 7:

For any symmetric matrices A and B, show that

(i) $(AB + BA)$ is symmetric.

(ii) $(AB - BA)$ is skew symmetric.

Solution :

Since A and B are symmetric,

$A' = A$ and $B' = B$ ---- (1)

(i) Consider,

$$(AB + BA)' = (AB)' + (BA)' \qquad (\because\ (A+B)' = A' + B')$$

$$= B'\,A' + A'\,B' \qquad \text{(by Reversal law)}$$

$$= BA + AB \qquad \text{(by (1))}$$

i.e., $(AB + BA)' = AB + BA$

$\Rightarrow$ $(AB + BA)$ is symmetric.

(ii) Consider $(AB - BA)' = (AB)' - (BA)'$

$$= B'\,A' - A'\,B' \qquad \text{(by Reversal law)}$$

$$= BA - AB \qquad \text{(by (1))}$$

i.e., $(AB - BA)' = -(AB - BA)$ (by taking minus sign common)

$\Rightarrow$ $(AB - BA)$ is skew-symmetric.

Example 8:

Show that $\begin{bmatrix} 3 & 1-i & 2+i \\ 1+i & 7 & -i \\ 2-i & i & 5 \end{bmatrix}$ is Hermitian matrix.

Solution :

$$\text{Let} \quad A = \begin{bmatrix} 3 & 1-i & 2+i \\ 1+i & 7 & -i \\ 2-i & i & 5 \end{bmatrix} \qquad \text{---- (1)}$$

$$\Rightarrow \quad \overline{A} = \begin{bmatrix} 3 & 1+i & 2-i \\ 1-i & 7 & i \\ 2+i & -i & 5 \end{bmatrix}$$

$$\Rightarrow \quad (\overline{A})' = \begin{bmatrix} 3 & 1-i & 2+i \\ 1+i & 7 & -i \\ 2-i & +i & 5 \end{bmatrix}$$

$$\Rightarrow \quad A^* = \begin{bmatrix} 3 & 1-i & 2+i \\ 1+i & 7 & -i \\ 2-i & i & 5 \end{bmatrix}$$

From (1) and (2), $A^* = A$.

$\Rightarrow$ A is Hermitian matrix.

Example 9:

Show that $\begin{bmatrix} 0 & 5-i & 3+2i \\ -5-i & i & 1-i \\ -3+2i & -1-i & -i \end{bmatrix}$ is skew-Hermitian matrix.

Solution :

$$\text{Let} \quad A = \begin{bmatrix} 0 & 5-i & 3+2i \\ -5-i & i & 1-i \\ -3+2i & -1-i & -i \end{bmatrix}$$

$$\Rightarrow \quad \overline{A} = \begin{bmatrix} 0 & 5+i & 3-2i \\ -5+i & -i & 1+i \\ -3-2i & -1+i & i \end{bmatrix}$$

$$(\overline{A})' = \begin{bmatrix} 0 & -5+i & -3-2i \\ 5+i & -i & -1+i \\ 3-2i & 1+i & i \end{bmatrix}$$

$$A^* = \begin{bmatrix} 0 & -5+i & -3-2i \\ 5+i & -i & -1+i \\ 3-2i & 1+i & i \end{bmatrix}$$

$$\Rightarrow \quad -A^* = \begin{bmatrix} 0 & 5-i & 3+2i \\ -5-i & i & 1-i \\ -3+2i & -1-i & -i \end{bmatrix}$$

From (1) and (2), $A = -A^*$

or $A^* = -A$

$\Rightarrow$ A is skew-Hermitian.

***Example* 10:**

Show that the sum of two Hermitian matrices is Hermitian matrix.

Solution :

Let A and B be Hermitian matrices.

$\Rightarrow \quad A^* = A \text{ and } B^* = B$ ---- (1)

To show that $(A+B)$ is Hermitian matrix, consider,

$$(A+B)^* = \left(\overline{A+B}\right)' \qquad \left(\because\ A^* = \left(\overline{A}\right)'\right)$$

$$= \left(\overline{A}+\overline{B}\right)' \qquad \left(\because\ \overline{A+B} = \overline{A}+\overline{B}\right)$$

$$= \left(\overline{A}\right)' + \left(\overline{B}\right)' \qquad \left(\because\ (A+B)' = A' + B'\right)$$

$$= A^* + B^*$$

i.e., $(A+B)^* = A + B$ (by (1))

$\Rightarrow$ $(A+B)$ is Hermitian matrix.

***Example* 11:**

For any square matrix A, show that $A\,A^*$ and A^*A are Hermitian matrices.

Solution :

To show that $(A\,A^*)$ is Hermitian, consider

$$(A\,A^*)^* = (A^*)^*\,A^* \qquad \text{(Reversal law)}$$

$$= A\cdot A^* \qquad (\because (A^*)^* = A)$$

i.e., $(A\,A^*)^* = AA^*$

$\Rightarrow$ AA^* is a Hermitian matrix.

Similarly it can be easily shown that A^*A is Hermitian (left as an excercise).

***Example* 12:**

If A and B are Hermitian matrices then AB is Hermitian if and only if $AB = BA$.

Solution :

Since A and B are Hermitian

$A^* = A$ and $B^* = B$ ---- (1)

First suppose that AB is Hermitian

$\Rightarrow \quad (AB)^* = AB$

or $AB = (AB)^*$

$= B^* A^*$ (Reversal la

i.e., $AB = BA$ (by (1 ---- (2

Conversely suppose that $AB = BA$

To show that AB is Hermitian, consider,

$(AB)^* = B^* A^*$ (Reversal la

$= BA$ (by (1

i.e., $(AB)^* = AB$ (by (2

Hence AB is Hermitian.

Example 13:

For any skew-Hermitian matrices A and B, show that $(A + B)$ is also skew-Hermitian.

Solution :

Since A and B are skew-Hermitian,

$$A^* = -A \text{ and } B^* = -B \qquad \text{---- (1)}$$

To show that $(A + B)$ is skew-Hermitian, consider,

$(A + B)^* = A^* + B^*$

$= -A - B$ (by (1))

i.e., $(A + B)^* = -(A + B)$

Hence $(A + B)$ is also skew-symmetric.

Example 14:

For any Hermitian matrices A and B show that

i) $AB + BA$ is Hermitian and

ii) $AB - BA$ is skew-Hermitian.

Solution :

Since A and B are Hermitian matrices

$$\Rightarrow \quad A^* = A \text{ and } B^* = B \qquad \text{---- (1)}$$

(i) To show that $(AB + BA)$ is Hermitian, consider

$(AB + BA)^* = (AB)^* + (BA)^*$ $(\because (A+B)^* = A^* + B^*)$

$= (B^* A^*) + (A^* B^*)$ (Reversal law)

i.e., $(AB + BA)^* = BA + AB$ (by (1))

Thus $(AB + BA)$ is Hermitian.

(ii) To show that $(AB - BA)$ is skew - Hermitian, consider

$$(AB - BA)^* = (AB)^* - (BA)^* \qquad (\because (A-B)^* = A^* - B^*)$$

$$= B^*A^* - A^*B^* \qquad \text{(Reversal law)}$$

$$= BA - AB \qquad \text{(by (1))}$$

i.e., $(AB - BA)^* = -(AB - BA)$.

Hence $(AB - BA)$ is skew-Hermitian matrix.

Example 15:

For any square matrix, show that $A + A^*$, AA^* are Hermitian.

Solution :

To show that $A + A^*$ is Hermitian, consider

$$(A + A^*)^* = A^* + (A^*)^* \qquad (\because (A+B)^* = A^* + B^*)$$

$$= A^* + A \qquad (\because (A^*)^* = A)$$

i.e., $(A + A^*)^* = A + A^*$ $\qquad (\because A + B = B + A)$

Thus $(A + A)^*$ is Hermitian.

Further to show that AA^* is Hermitian, consider.

$$(A A^*)^* = (A^*)^* A^* \qquad (\because (AB)^* = B^* A^*)$$

i.e., $(A A^*)^* = AA^*$

Hence AA^* is Hermitian.

1.12 Orthogonal matrix

Any square matrix A is said to be **orthogonal** if

$A A' = A' A = I$, where I is identity matrix.

Note :

1. Since $AA^{-1} = A^{-1}A = I$, the above definition reveals that $A^{-1} = A'$.
   i.e., If A is orthogonal, its inverse is same as its transpose.
2. If A is orthogonal then A' is also orthogonal.

Theorem : If A and B are orthogonal matrices of order n, then AB and BA are also orthogonal.

Proof : Since A is orthogonal, $AA' = A' A = I$ ---- (1)

also since B is orthogonal, $= BB' = B' B = I$ ---- (2)

To show that AB is orthogonal :

Consider $(AB)(AB)' = (AB)(B'A')$ (Reversal law)

$= A\,(BB')\,A'$ (by (2)

$= A \cdot I A'$

$= AA'$ (by (1)

i.e., $(AB)\,(AB)' = I$ (Reversal law

Similarly $(AB)'\,(AB) = (B'A')\,(AB)$

$= B'\,(A'\,A)B$ (by (1)

$= B' I B$ (by (2)

$= B' B$

i.e., $(AB)'\,(AB) = I$

i.e., $(AB)\,(AB)' = (AB)'\,(AB) = I$. Hence AB is orthogonal.

Similarly it can be easily proved that BA is orthogonal (left as an exercise).

1.13 Unitary matrix

Any square matrix A is said to be **unitary** if

$$A^* A = I$$

Note :

1. If A is unitary $A^{-1} = A^*$
2. If A is unitary then A^* is also unitary.

Theorem : If A and B are unitary matrices of the same order then AB and BA are also unitary.

Proof : Since A and B are unitary

$A^* A = I$ ---- (1)

and $B^* B = I$ ---- (2)

To show that AB is unitary, consider

$(AB)^*\,(AB) = (B^*\,A^*)\,(AB)$ (Reversal law)

$= B^*\,(A^*\,A)B$

$= B^*\,IB$

$= B^*\,B$ (by (1))

$= I$

$\therefore$ $(AB)^*\,(AB) = I$ (by (2))

Hence AB is unitary.

Similarly it can be proved that BA is unitary. (left as an exercise).

Worked Example

***Example* 1:**

Show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal

Solution :

$$\text{Let} \quad A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\Rightarrow \quad A' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\text{Consider} \quad A'A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e., $A'A = I$

Hence A is orthogonal matrix.

***Example* 2:**

Show that $\frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$ is orthogonal.

Solution :

$$\text{Let} \quad A = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} \Rightarrow A' = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A'A = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1+4+4 & -2-2+4 & -2+4-2 \\ -2-2+4 & 4+1+4 & 4-2-2 \\ -2+4-2 & 4-2-2 & 4+4+1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A' A = I.$

Hence given matrix is orthogonal.

***Example* 3:**

Show that $\frac{1}{2\sqrt{2}}\begin{bmatrix} 1+i & \sqrt{3}+i\sqrt{3} \\ \sqrt{3}+i\sqrt{3} & -1-i \end{bmatrix}$ is unitary.

Solution :

Let $A = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1+i & \sqrt{3}+i\sqrt{3} \\ \sqrt{3}+i\sqrt{3} & -1-i \end{bmatrix}$

$\Rightarrow \quad \overline{A} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1-i & \sqrt{3}-i\sqrt{3} \\ \sqrt{3}-i\sqrt{3} & -1+i \end{bmatrix}$

$\Rightarrow \quad (\overline{A})' = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1-i & \sqrt{3}-i\sqrt{3} \\ \sqrt{3}-i\sqrt{3} & -1+i \end{bmatrix} = A^*$

Consider,

$$A^* A = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1-i & \sqrt{3}-i\sqrt{3} \\ \sqrt{3}-i\sqrt{3} & -1+i \end{bmatrix} \frac{1}{2\sqrt{2}}\begin{bmatrix} 1+i & \sqrt{3}+i\sqrt{3} \\ \sqrt{3}+i\sqrt{3} & -1-i \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 1+1+3+3 & \sqrt{3}+\sqrt{3}-\sqrt{3}-\sqrt{3} \\ \sqrt{3}+\sqrt{3}-\sqrt{3}-\sqrt{3} & 3+3+1+1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e., $A^* A = I$

Hence A is a unitary matrix.

Example 4 :

Show that $\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ is an orthogonal matrix, provided (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) are direction cosines of three mutually perpendicular lines.

Solution :

$$\text{Let} \quad A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

$$\Rightarrow \quad A' = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$$

Consider,

$$A'A = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

$$= \begin{bmatrix} l_1^2 + l_2^2 + l_3^2 & l_1m_1 + l_2m_2 + l_3m_3 & l_1n_1 + l_2n_2 + l_3n_3 \\ l_1m_1 + l_2m_2 + l_3m_3 & m_1^2 + m_2^2 + m_3^2 & m_1n_1 + m_2n_2 + m_3n_3 \\ l_1n_1 + l_2n_2 + l_3n_3 & m_1n_1 + m_2n_2 + m_3n_3 & n_1^2 + n_2^2 + n_3^2 \end{bmatrix} \text{---- (1)}$$

If (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) are the dc's of three mutually perpendicular lines, we have

$$l_1^2 + l_2^2 + l_3^2 = \sum_{i=1}^{3} l_i^2 = 1 = \sum_{i=1}^{3} m_i^2 = \sum_{i=1}^{3} n_i^2$$

$$l_1m_1 + l_2m_2 + l_3m_3 = \sum_{i=1}^{3} l_i\, m_i = 0 = \sum_{i=1}^{3} l_i\, n_i = \sum_{i=1}^{3} m_i\, n_i$$

$$\therefore \quad (1) \Rightarrow A'A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e., $A'A = I$

Hence A is an orthogonal matrix

$$= \frac{1}{9}\begin{bmatrix} 1+4+4 & -2-2+4 & -2+4-2 \\ -2-2+4 & 4+1+4 & 4-2-2 \\ -2+4-2 & 4-2-2 & 4+4+1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A'A = I.$$

Hence given matrix is orthogonal.

***Example* 3:**

Show that $\frac{1}{2\sqrt{2}}\begin{bmatrix} 1+i & \sqrt{3}+i\sqrt{3} \\ \sqrt{3}+i\sqrt{3} & -1-i \end{bmatrix}$ is unitary.

Solution :

Let $\quad A = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1+i & \sqrt{3}+i\sqrt{3} \\ \sqrt{3}+i\sqrt{3} & -1-i \end{bmatrix}$

$$\Rightarrow \quad \overline{A} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1-i & \sqrt{3}-i\sqrt{3} \\ \sqrt{3}-i\sqrt{3} & -1+i \end{bmatrix}$$

$$\Rightarrow \quad (\overline{A})' = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1-i & \sqrt{3}-i\sqrt{3} \\ \sqrt{3}-i\sqrt{3} & -1+i \end{bmatrix} = A^*$$

Consider,

$$A^*A = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1-i & \sqrt{3}-i\sqrt{3} \\ \sqrt{3}-i\sqrt{3} & -1+i \end{bmatrix}\frac{1}{2\sqrt{2}}\begin{bmatrix} 1+i & \sqrt{3}+i\sqrt{3} \\ \sqrt{3}+i\sqrt{3} & -1-i \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 1+1+3+3 & \sqrt{3}+\sqrt{3}-\sqrt{3}-\sqrt{3} \\ \sqrt{3}+\sqrt{3}-\sqrt{3}-\sqrt{3} & 3+3+1+1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e., $A^*A = I$

Hence A is a unitary matrix.

***Example* 4 :**

Show that $\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ is an orthogonal matrix, provided (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) are direction cosines of three mutually perpendicular lines.

Solution :

Let $A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$

$\Rightarrow \quad A' = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$

Consider,

$$A'A = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

$$= \begin{bmatrix} l_1^2+l_2^2+l_3^2 & l_1m_1+l_2m_2+l_3m_3 & l_1n_1+l_2n_2+l_3n_3 \\ l_1m_1+l_2m_2+l_3m_3 & m_1^2+m_2^2+m_3^2 & m_1n_1+m_2n_2+m_3n_3 \\ l_1n_1+l_2n_2+l_3n_3 & m_1n_1+m_2n_2+m_3n_3 & n_1^2+n_2^2+n_3^2 \end{bmatrix} \quad \text{---- (1)}$$

If (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) are the dc's of three mutually perpendicular lines, we have

$$l_1^2+l_2^2+l_3^2 = \sum_{i=1}^{3} l_i^2 = 1 = \sum_{i=1}^{3} m_i^2 = \sum_{i=1}^{3} n_i^2$$

$$l_1m_1+l_2m_2+l_3m_3 = \sum_{i=1}^{3} l_i\, m_i = 0 = \sum_{i=1}^{3} l_i\, n_i = \sum_{i=1}^{3} m_i\, n_i$$

$$\therefore \quad (1) \Rightarrow A'A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e., $A'A = I$

Hence A is an orthogonal matrix

Exercise

1. If $A = \begin{bmatrix} 2 & 4 & 7 \\ 4 & 0 & 1 \\ 7 & 1 & 3 \end{bmatrix}$ show that A is symmetric

2. Show that $\begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & -2 \\ -3 & 2 & 0 \end{bmatrix}$ is skew symmetric

3. If A is a symmetric matrix show that kA is also symmetric (k being a constant).
4. If A is skew-symmetric and k is any positive constant then kA is also skew-symmetric.
5. Show that (i) $\begin{bmatrix} 2 & i & 1+i \\ -i & 7 & 2-i \\ 1-i & 2+i & 6 \end{bmatrix}$, (ii) $\begin{bmatrix} 0 & 5-i & -i \\ 5+i & 1 & i \\ i & -i & 4 \end{bmatrix}$ are Hermitian matrices.
6. Show that (i) $\begin{bmatrix} i & -i & 2+i \\ -i & 0 & 1+2i \\ -2+i & -1+2i & -i \end{bmatrix}$, (ii) $\begin{bmatrix} 0 & -4-2i & 1-i \\ 4-2i & -i & 1-2i \\ -1-i & -1-2i & i \end{bmatrix}$ are skew-Hermitia matrices.
7. If A is Hermitian matrix and k is a real constant then show that (kA) is Hermitian.
8. For any square matrix A, show that $A^* A$ is Hermitian.
9. For any square matrix A show that $A - A^*$ is skew-Hermitian.
10. For any Hermitian matrix A show that $(A+\overline{A})$ is symmetric and $(A-\overline{A})$ is skew-symmetr
11. Show that $\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ is orthogonal.
12. Show that $\frac{1}{6}\begin{bmatrix} -2 & 4 & 4 \\ 4 & -2 & 4 \\ 4 & 4 & -2 \end{bmatrix}$ is orthogonal.
13. Show that $\begin{bmatrix} i & 2i \\ 2i & -i \end{bmatrix}$ is a unitary matrix.
14. Show that $\begin{pmatrix} -i & i \\ i & i \end{pmatrix}$ is a unitary matrix.

1.14 Determinants

A real valued function from the set of square matrices to the set of real numbers is called a **Determinant**.

In other words a unique real value associated with every square matrix is called the determinant. If A is a given square matrix then its determinant is expressed by keeping its elements within a pair of vertical bars & symbolically it is denoted as $|A|$ (or Δ).

If A is of order $n \times n$ then $|A|$ is said to be of order n.

Method of evaluation of a determinant

If $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ then $|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ then $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Note:

(1) The above expansion of 3rd order determinant is along the I^{st} row. We can expand it along any row or column by following the sign convention $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$.

2. The 4th order determinant is evaluated by expanding it interms of four 3rd order determinants.

1.15 Rank of a matrix

If A is $m \times n$ matrix, then a determinant of a square sub-matrix of order $r \times r$ obtained by retaining only r-number of rows and columns of A is called a **minor** of A of **order** r.

Note: Each element of A is a minor of order 1.

Definition: A matrix A is said to be of rank r if

i) there exists atleast one non-zero minor of order r.

ii) all the minors of order greater than r are zero.

In otherwords, the order of a largest non-zero minor of A is called the rank of A. The rank of A is denoted by $\rho(A)$.

1.16 Elementary transformations (Elementary Operations)

The following three operations called as elementary row/column transformations can be performed on a given matrix.

(i) Any two rows (or columns) can be interchanged ($R_i \leftrightarrow R_j$).

(ii) Every element of any row (or column) can be multiplied by a non-zero cons (i.e., $R_i \to k.R_i$)

(iii) Every element of any row (or column) can be added/subtracted with k-times the correspon element of any other row ($R_i \to R_i \pm kR_j$)

Equivalent matrices

If B is a matrix obtained by applying the elementary operations on the given matrix A A and B are called as the equivalent matrices & denoted $A \sim B$.

Echelon form

A matrix is said to be in echelon form, if

i) the first non-zero element in any row is unity.

ii) all the elements below this leading non-zero entry are zero.

iii) the number of zeros preceeding the leading non-zero entry in any row is less than the nur of such zeros in the succeeding rows.

iv) all the non-zero rows must appear above the zero rows.

For example:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Normal form

A matrix of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} I_r & 0 \end{bmatrix}$ or $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ or $[I_r]$, where I_r is an identity matrix of o $r \times r$ is said to be in the normal form.

For example :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Properties:

1. $\rho(A) = 0$ if A is a null matrix.
2. $\rho(A) \geq 1$, if A is a non-zero matrix.

3. $\rho(A) \le r$ when every minor of order $(r+1)$ vanishes.
4. $\rho(A) \ge r$ when $\exists$ a non-zero minor of order r.
5. $\rho(A) \le \min(m, n)$, if A is of order $m \times n$.
6. Rank of transpose of A is same as rank of A. i.e., $\rho(A') = \rho(A)$.
7. For any two matrices A & B, $\rho(AB) \le \rho(A)$ and $\rho(AB) \le \rho(B)$.
8. Rank of a matrix remains unchanged even when it is multiplied by a non-singular matrix.
9. Rank of a matrix remains unchanged even after applying elementary transformation. (i.e., rank of equivalent matrices is same).
10. If a matrix is reducible to the echelon form containing r number of non-zero rows then its rank is r.
11. If a matrix is reducible to the normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ then its rank is r.

Note:

To find the rank of a matrix the properties (9), (10) or (11) can be conviniently used. (instead of proceeding to find minors directly).

WORKED EXAMPLES

Example : 1

Find the rank of $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}$

Solution :

Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & -1 & -4 \end{bmatrix} \qquad \begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{pmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \to R_3$$

Clearly, $|A| = 0$ which is a minor of order 3.

$\therefore \quad \rho(A) < 3$

Further, there is a minor $\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = -2 \neq 0$

$\therefore \quad \rho(A) = 2$ (since order of above non-zero minor is

Note: In the above example the final form of A is echalon with two non-zero rows & he $\rho(A) = 2$

Example : 2

Find $\rho(A)$, if $A = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & 0 & 1 & -2 \end{bmatrix}$

Solution :

$$A \sim \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad \begin{pmatrix} R_3 \to R_3 - R \\ R_4 \to R_4 - R \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \to R_3 - 3R_2 \,;\, R_4 \to R_4 -$$

— is in the echelon form & contains 2 non-zero rows. Hence $\rho(A) = 2$.

Example : 3

Find the rank of $\begin{bmatrix} 1 & 5 & 4 \\ 2 & 8 & 6 \\ 3 & 22 & 7 \end{bmatrix}$

Solution :

Let $A = \begin{bmatrix} 1 & 5 & 4 \\ 2 & 8 & 6 \\ 3 & 22 & 7 \end{bmatrix}$

$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & -2 & -2 \\ 0 & 7 & -5 \end{bmatrix}$ $(R_2 \rightarrow R_2 - 2R_1 \,;\, R_3 \rightarrow R_3 -$

$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & 1 & 1 \\ 0 & 7 & -5 \end{bmatrix}$ $(R_2 \rightarrow -\frac{1}{2}$

$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 5 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & -12 \end{bmatrix}$ $(R_3 \rightarrow R_3 -$

This is in echelon form & contains 3 non-zero rows

$\Rightarrow \ \rho(A) = 3.$

Example : 4

Find $\rho(A)$ for $A = \begin{bmatrix} 1 & 3 & 4 & 1 \\ 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \end{bmatrix}$

Solution :

$A \sim \begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $(R_2 \rightarrow R_2 - R_1 \,;\, R_3 \rightarrow R_3 -$

This is in echelon form & contains 2 non-zero rows.

$\Rightarrow \quad \rho(A) = 2.$

Example : 5

Reduce the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix}$ to the echelon form and hence find its rank.

Solution :

Let $$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -1 & -2 \end{bmatrix} \qquad \begin{pmatrix} R_2 \to R_2 - 3 \\ R_3 \to R_3 - 2 \end{pmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \to 2R_3 -$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \left(R_2 \to -\frac{1}{2}\right.$$

This is in the echelon form and contains two non-zero rows

Hence $\rho(A) =$

Example : 6

Reduce $\begin{bmatrix} 1 & 4 & 3 & 2 \\ 3 & 2 & 1 & 4 \\ 4 & 6 & 4 & 6 \\ 7 & 8 & 5 & 10 \end{bmatrix}$ to the echelon form and hence find the rank.

Solution :

Let $$A = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 3 & 2 & 1 & 4 \\ 4 & 6 & 4 & 6 \\ 7 & 8 & 5 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 3 & 2 \\ 3 & 2 & 1 & 4 \\ 1 & 4 & 3 & 2 \\ 3 & 2 & 1 & 4 \end{bmatrix} \qquad \begin{pmatrix} R_3 \to R_3 - \\ R_4 \to R_4 - \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 3 & 2 \\ 3 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{pmatrix} R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_2 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & -10 & -8 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (R_2 \to R_2 - 3R_1)$$

i.e., $$A \sim \begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & 1 & 4/5 & 1/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left(R_2 \to -\frac{1}{10} R_2\right)$$

This is in the echelon form and contains two non-zero rows. Hence $\rho(A) = 2$.

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xample : 7

Reduce $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ into echelon form and hence find its rank.

olution :

Let $$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \qquad \begin{matrix} (R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1) \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \to R_3 - \quad R_4 \to R_4 -$$

This is in the echelon form and contains 2 non-zero rows.

Hence $\rho(A) = 2$

Example : 8

Find the rank of $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 4 \\ 2 & 1 & 5 \end{bmatrix}$ by reducing to normal form

Solution :

$$\text{Let} \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 4 \\ 2 & 1 & 5 \end{bmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & -3 & 3 \end{bmatrix}$$

$(R_2 \to R_2 - 2R_1 \,;\, R_3 \to R_3 - 3R_1 \,;\, R_4 \to R_4 - 2$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \to R_3 - R_2 \,;\, R_4 \to R_4 - 3$$

Now we apply column operations

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (c_2 \to c_2 - 2c_1 \,;\, c_3 \to c_3 -$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (c_2 \to c_2 + c_2)$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (c_2 \to - c_2$$

$$\Rightarrow \quad A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence $\rho(A) = 2$

Example : 9

Reduce the matrix $\begin{bmatrix} -8 & -1 & -3 & 4 \\ 0 & 3 & 2 & 2 \\ 8 & 1 & 3 & 6 \end{bmatrix}$

Solution :

Let $A = \begin{bmatrix} -8 & -1 & -3 & 4 \\ 0 & 3 & 2 & 2 \\ 8 & 1 & 3 & 6 \end{bmatrix}$

$$\sim \begin{bmatrix} -8 & -1 & -3 & 4 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \qquad (R_3 \to R_3 + R$$

$$\sim \begin{bmatrix} 1 & -1 & 4 & -3 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 10 & 0 \end{bmatrix} \qquad \left(C_1 \to -\frac{1}{8} C \right. \quad (C_3 \leftrightarrow C_4)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 10 & 0 \end{bmatrix} \qquad \begin{matrix} (C_2 \to C_2 + C \\ C_3 \to C_3 - 4C \\ C_4 \to C_4 + 3C \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

$\left(C_2 \rightarrow \frac{1}{3}\right.$

$C_3 \rightarrow \frac{1}{2}$

$C_4 \rightarrow \frac{1}{2}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

$(C_3 \leftrightarrow C_3 - C_2;\ C_4 \rightarrow C_4 -$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\left(C_3 \rightarrow \frac{1}{5}\right.$

$$\sim [I_3\ 0]$$

This is in normal form.

Hence $\rho(A) = 3$.

Example : 10

Find the rank of $\begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$ by reducing into normal form.

Solution :

$$\text{Let} \quad A = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 6 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$(R_2 \rightarrow R_2 - R$

$R_3 \rightarrow R_3 - 3R$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{pmatrix} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 + R_2 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 + 2C_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{pmatrix} C_3 \to C_3 + 3C_2 \\ C_4 \to C_4 + C_2 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_2 \to -C_2$$

$$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

This is in normal form and hence $\rho(A) = 2$.

Example : 11

Find the rank of $\begin{bmatrix} 1 & 1 & 2 & -3 \\ 4 & -1 & 0 & 2 \\ 0 & -3 & 0 & 4 \\ 0 & -1 & 0 & 2 \end{bmatrix}$ by reducing into normal form.

Solution :

Let $\quad A = \begin{bmatrix} 1 & 1 & 2 & -3 \\ 4 & -1 & 0 & 2 \\ 0 & -3 & 0 & 4 \\ 0 & -1 & 0 & 2 \end{bmatrix}$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 2 & -3 \\ 0 & -5 & -8 & 14 \\ 0 & -3 & 0 & 4 \\ 0 & -1 & 0 & 2 \end{bmatrix}$$

$(R_2 \to R_2 - 4$

$$\sim \begin{bmatrix} 1 & 2 & 1 & -3 \\ 0 & -8 & -5 & 14 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$(C_2 \leftrightarrow$

$$\sim \begin{bmatrix} 1 & 2 & 1 & -3 \\ 0 & -8 & -5 & 14 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$(R_4 \to 3R_4 -$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -8 & -5 & 14 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$(C_2 \to C_2 - 2$

$C_3 \to C_3 - C$

$C_4 \to C_4 + 3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 7 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\left(C_2 \to -\frac{1}{8}\right.$

$C_4 \to \frac{1}{2}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$(C_3 \to C_3 + 5$

$C_4 \to C_4 - 7$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to -\frac{1}{3}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad C_4 \to C_4 - 2C_3$$

i.e., $A \sim [I_4]$

This is the required normal form.

Hence $\rho(A) = 4$.

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xample : 12

Find the value of a such that the rank of

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 4 & 4 & 1 & -3 \\ 2 & a & 2 & 2 \\ 9 & 9 & 3 & a \end{bmatrix} \text{ is } 3$$

olution :

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 4 & 4 & 1 & -3 \\ 2 & a & 2 & 2 \\ 9 & 9 & 3 & a \end{bmatrix}$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & (a-2) & 0 & 4 \\ 0 & 0 & 0 & (a+9) \end{bmatrix}$$

$$(R_2 \to R_2 - 4R_1\ ;\ R_3 \to R_3 - 2R_1\ ;\ R_4 \to R_4 - 9R_1)$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & (a-2) & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & (a+9) \end{bmatrix} \qquad (R_2 \leftrightarrow R_3)$$

Since $\rho(A) = 3$, there must be only 3 non-zero rows

$\Rightarrow \quad a + 9 = 0$

$\Rightarrow \quad a = -9.$

Exercise

1. Find the rank of the following by reducing into echelon form.

i) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & -2 \\ 4 & 2 & -3 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & 5 \\ -1 & 4 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 5 & 6 & 11 & 16 \\ 6 & 7 & 12 & 17 \\ 7 & 8 & 13 & 18 \\ 8 & 9 & 14 & 19 \end{bmatrix}$

iv) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ v) $\begin{bmatrix} 1 & 2 & 4 \\ 4 & 8 & 16 \\ 2 & 5 & 16 \end{bmatrix}$ vi) $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

vii) $\begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ viii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ ix) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 2 & 1 & 3 \\ 2 & 4 & 3 & 2 \\ 6 & 1 & 7 & 5 \end{bmatrix}$

2. Find the rank by reducing into normal form

i) $\begin{bmatrix} 1 & 2 & 6 & 7 \\ 1 & 3 & 4 & 5 \\ 1 & 5 & 0 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 2 & -2 \\ -1 & 3 & 3 & 6 \\ 4 & 4 & 4 & -7 \end{bmatrix}$

iv) $\begin{bmatrix} -1 & 2 & 2 \\ 3 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ v) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ vi) $\begin{bmatrix} 1 & 0 & 8 & 1 \\ 8 & 0 & 0 & 1 \\ 0 & 0 & 1 & 8 \\ 0 & 8 & 1 & 8 \end{bmatrix}$

vii) $\begin{bmatrix} 4 & 2 & 6 & -1 \\ 6 & 1 & 3 & 8 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ viii) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ -2 & -1 & -1 & -1 \\ 2 & 3 & 0 & 1 \end{bmatrix}$ ix) $\begin{bmatrix} 3 & 4 & 1 & 2 \\ 1 & 2 & 0 & -1 \\ -2 & 3 & 2 & 5 \end{bmatrix}$

x) $\begin{bmatrix} 2 & 1 & 3 & 2 \\ 4 & 0 & 2 & 6 \\ 0 & 1 & 2 & -2 \end{bmatrix}$ xi) $\begin{bmatrix} 1 & 3 & -3 & -4 \\ 1 & -1 & 3 & -6 \\ 5 & 3 & 5 & 11 \end{bmatrix}$ xii) $\begin{bmatrix} 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 4 & 6 \end{bmatrix}$

xiii) $\begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

3. Show that the rank of $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is 1.

Answers

1. i) 2 ii) 2 iii) 2 iv) 2 v) 3
   vi) 3 vii) 2 viii) 1 ix) 4
2. i) 2 ii) 3 iii) 3 iv) 2 v) 1
   vi) 4 vii) 2